A Glimpse Into Operator Theory
Part II

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From Last Week...

Proposition

$L^2_a(\mathbb{D})$ is a Hilbert space.

Sketch proof.

It suffices to show that L^2_a is a closed subset of $L^2\left(\mathbb{D}, \frac{dA}{\pi}\right)$. We need the following fact from Complex Analysis:

$$|f(w)| \leq \frac{1}{1-|w|^2} \, ||f||_{L^2_a} \, .$$

1. Let $\{f_n\} \subset L^2_a(\mathbb{D})$ and assume $f_n \to f \in L^2\left(\mathbb{D}, \frac{dA}{\pi}\right)$ in norm.
2. Fix $K \subseteq \mathbb{D}$ compact. $|f_n(z) - f_m(z)| \leq \frac{1}{\mathrm{dist}(K, \partial\mathbb{D})} \, ||f_n - f_m||$. f_n is uniformly Cauchy on compact sets.
3. f_n converges uniformly on K. Morera's Theorem says f_n converges to an analytic function.

Outline

Operator Algebras

Definition

A **complex algebra** *is a vector space k over* $\mathbb{C}$ *with a multiplication satisfying:*

1. $A(BC) = (AB)C$
2. $(A + B)C = AC + BC$
3. $A(B + C) = AB + AC$
4. $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

A **Banach algebra** $\mathfrak{A}$ *is a Banach space which is a complex algebra with*

$$||AB|| \leq ||A||\,||B||$$

Definition

An **involution** $*: \mathfrak{A} \to \mathfrak{A}$ *is an operation satisfying:*

1. $(A^*)^* = A$
2. $(AB)^* = B^* A^*$
3. $(\alpha A + B)^* = \overline{\alpha} A^* + B^*$.

A $*$-**algebra** *is a Banach algebra* $\mathfrak{A}$ *with an involution, called the adjoint.*

$\mathfrak{A}$ *is called* **unital** *if there exists an element* $I \in \mathfrak{A}$ *such that* $IA = AI = A$ *for all* $A \in \mathfrak{A}$. $\mathfrak{A}$ *is called* **commutative** *if* $AB = BA$ *for all* $A, B \in \mathfrak{A}$.

Definition

A C^*-**algebra** $\mathfrak{A}$ *is a* $*$*-algebra satisfying for all* $A \in \mathfrak{A}$
$$||A^*A|| = ||A||^2 .$$

Proposition

A $*$*-algebra* $\mathfrak{A}$ *satisfying* $||A^*A|| \geq ||A||^2$ *for all* $A \in \mathfrak{A}$ *is a* C^**-algebra.*

Proof

1. $||A||^2 \leq ||A^*A|| \leq ||A^*||\,||A||$. *So* $||A|| \leq ||A^*||$.
2. *Replacing* A *with* A^*, *we have* $||A^*|| \leq ||A^{**}|| = ||A||$.
3. *So* $||A|| = ||A^*||$ *and* $||A^*A|| \leq ||A^*||\,||A|| = ||A||^2$.

Simple Example: $\mathbb{C}$

1. $\mathbb{C}$ is a complex algebra over itself.
2. For $z, w \in \mathbb{C}$, $|zw| = |z|\,|w|$. So $\mathbb{C}$ is a unital commutative Banach algebra.
3. The involution on $\mathbb{C}$ is $z^* = \overline{z}$. So $\mathbb{C}$ is a $*$-algebra.
4. We have $|z^*z| = |z^*|\,|z| = |z|\,|z| = |z|^2$, so $\mathbb{C}$ is a C^*-algebra.

Bounded linear operators on $\mathcal{H}$

For a Hilbert space $\mathcal{H}$

1. $\mathscr{B}(\mathcal{H})$ is a Banach space.
2. Define $AB(h) = (A \circ B)(h)$. $\mathscr{B}(\mathcal{H})$ into a complex algebra.
3. By the properties of the norm, $||AB|| \leq ||A||\,||B||$. So $\mathscr{B}(\mathcal{H})$ is a Banach algebra.
4. The adjoint map $* : \mathscr{B}(\mathcal{H}) \to \mathscr{B}(\mathcal{H})$ mapping $A \mapsto A^*$ is an involution. So $\mathscr{B}(\mathcal{H})$ is a $*$-algebra.
5. We have $||A^*A|| = ||A||^2$. So $\mathscr{B}(\mathcal{H})$ is a unital C^*-algebra.

Proposition

Suppose $\mathfrak{A}$ is is C^*-algebra. Every $A \in \mathfrak{A}$ can be written as $X + iY$ where X and Y are self-adjoint.

Proof

Let $X = \dfrac{A + A^*}{2}$ and $Y = \dfrac{i(A^* - A)}{2}$.

$$X + iY = \frac{A + A^*}{2} - \frac{A^* - A}{2} = A.$$

$$X^* = \frac{A^* + A}{2} = X$$

$$Y^* = \frac{\bar{i}(A^* - A)^*}{2} = \frac{i(A^* - A)}{2} = Y$$

Spectral Theory

Definition

For a unital Banach algebra $\mathfrak{A}$, $A \in \mathfrak{A}$ is **invertible** *if there exists $B \in \mathfrak{A}$ such that $AB = BA = I$.*

The **spectrum** *of A is*

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not invertible}\}.$$

The **point spectrum** *is the set of eigenvalues of A,*

$$\sigma_p(A) = \{\lambda \in \mathbb{C} : \ker(A - \lambda I) = 0\}.$$

Example

Let $\mathfrak{A} = C(Y)$ and $f \in \mathfrak{A}$.

$$\begin{aligned}\sigma(f) &= \{\lambda \in \mathbb{C} : f - \lambda \text{ is not invertible}\} \\ &= \{\lambda \in \mathbb{C} : \exists y \in Y \text{ st. } (f - \lambda)(y) = 0\} \\ &= f(Y).\end{aligned}$$

Lemma

Let $\mathfrak{A}$ be a unital Banach algebra. If $B \in \mathfrak{A}$ and $||I - B|| < 1$, then B is invertible.

Proof

1. *Let* $C = I - B$, *so* $||C|| = r < 1$. *Thus,* $||C^n|| \leq ||C||^n = r^n$.
2. *The partial sums converge so* $\displaystyle\sum_{n=0}^{\infty} ||C^n||$. *So* $Z = \sum_{n=0}^{\infty} C^n \in \mathfrak{A}$.
3. *Let* $Z_n = 1 + C + \cdots + C^n$. *So* $Z_n(I - C) = I - C^{n+1}$. *Since* $\left|\left|C^{n+1}\right|\right| \to 0$, $Z_n(I - C) \to Z(I - C) = I$.
4. *Similarly* $(I - C)Z = I$. *So* $I - C = I - (I - B) = B$ *is invertible.*

Theorem

Let $\mathfrak{A}$ be a unital Banach algebra and $A \in \mathfrak{A}$. Then $\sigma(A)$ is a non-empty compact subset of $\{\lambda \in \mathbb{C} : |\lambda| \leq ||A||\}$.

Proof

1. *Let $\lambda \in \mathbb{C}$ st. $|\lambda| > ||A||$. $\left|\left|I + \left(\frac{A}{\lambda} - I\right)\right|\right| = \left|\left|\frac{A}{\lambda}\right|\right| < 1$. So $\left(\frac{A}{\lambda} - I\right)$ is invertible. Thus $\lambda\left(\frac{A}{\lambda} - I\right) = A - \lambda I$ is invertible. So $\sigma(A) \subseteq \{\lambda : |\lambda| \leq ||A||\}$.*
2. *Fix $A \in \mathfrak{A}$. $\tau : \mathbb{C} \to \mathfrak{A}$ defined by $\tau(\lambda) = A - \lambda I$ is continuous. $\mathcal{G} = \{A \in \mathfrak{A} : A^{-1}\text{exists}\}$ is open, so $\tau^{-1}(\mathcal{G}) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is invertible}\} = \mathbb{C} \setminus \sigma(A)$ is open. So $\sigma(A)$ is closed.*
3. *$\sigma(A)$ is a closed and bounded subset of $\mathbb{C}$, so it is compact.*

Theorem (Gelfand-Mazur)

If $\mathfrak{A}$ is a unital Banach algebra which is a division algebra, then $\mathfrak{A}$ is isometrically isomorphic to $\mathbb{C}$.

Proof

1. For $A \in \mathfrak{A}$, $\sigma(A) \neq \emptyset$. So $\exists \lambda \in \sigma(A)$ st $A - \lambda I$ is not invertible.
2. $A - \lambda I = 0$ since $\mathfrak{A}$ is a division algebra. So, for all $A \in \mathfrak{A}$, $\exists \lambda_A \in \mathbb{C}$ such that $A = \lambda_A I$.
3. λ_A is unique. If $A = \lambda_A I = \lambda'_A I$, then $(\lambda_A - \lambda'_A) I = 0$. So $\lambda_A - \lambda'_A = 0$.
4. The map $\varphi(A) = \lambda_A$ is an isometry since
$$||\varphi(A)|| = |\lambda_A| = ||\lambda_A I|| = ||A|| \,.$$

Corollary

For $\mathfrak{A}$ a Banach algebra and $\mathcal{M} \subseteq \mathfrak{A}$ a maximal ideal, then $\mathfrak{A}/\mathcal{M} \cong \mathbb{C}$.

Ideals of Banach Algebras

Definition

A **complex homomorphism** *on a Banach algebra* $\mathfrak{A}$ *is a linear map* $\varphi : \mathfrak{A} \to \mathbb{C}$ *such that* $\varphi(AB) = \varphi(A)\varphi(B)$.

Remarks

1. *We usually will exclude* $\varphi \equiv 0$.
2. *If* $\mathfrak{A}$ *is a unital Banach algebra and* $\varphi : \mathfrak{A} \to \mathbb{C}$ *a complex homomorphism, then* $\varphi(I) = \varphi(II) = \varphi(I)\varphi(I)$. *Hence* $\varphi(I) = 1$.
3. *Let* A *be invertible in unital Banach algebra* $\mathfrak{A}$ *with complex homomorphism* $\varphi : \mathfrak{A} \to \mathbb{C}$.
$$1 = \varphi(I) = \varphi(AA^{-1}) = \varphi(A)\varphi(A^{-1}).$$
4. *From the previous remark, if* A *is invertible then* $\varphi(A) \neq 0$.

Theorem

Every complex homomorphism $\varphi \neq 0$ *of a unital Banach algebra* $\mathfrak{A}$ *is continuous and* $||\varphi|| = 1$.

Proof

1. *For* $A \in \mathfrak{A}$ *and* $\varphi(A) \neq 0$, $\varphi\left(I - \frac{A}{\varphi(A)}\right) = 0$. *So* $I - \frac{A}{\varphi(A)}$ *is not invertible.*
2. *So* $\left|\left|\frac{A}{\varphi(A)}\right|\right| \geq 1$. *Otherwise* $\left|\left|I - \left(I - \frac{A}{\varphi(A)}\right)\right|\right| < 1$ *which means* $I - \frac{A}{\varphi(A)}$ *is invertible.*
3. *So in this case* $||A|| \geq |\varphi(A)|$. *If* $\varphi(A) = 0$, *then* $||A|| \geq |\varphi(A)|$ *also. So we have* $||A|| \geq |\varphi(A)|$ *always.*
4. $|\varphi(A)| \leq ||A||$ *says* $|\varphi| \leq 1$. $\varphi(I) = 1$, *so* $||\varphi|| = 1$.

Definition

*A (**2-sided**) **ideal** $\mathfrak{J}$ in a Banach algebra $\mathfrak{A}$ is a subspace (not necessarily closed) of $\mathfrak{A}$ with the property $\forall A \in \mathfrak{A}, J \in \mathfrak{J}$, $AJ, JA \in \mathfrak{J}$.*

*If $\mathfrak{J} \subsetneq \mathcal{A}$ we say $\mathfrak{J}$ is a **proper ideal**.*

*If there is no ideal $\mathfrak{J}'$ st. $\mathfrak{J} \subsetneq \mathfrak{J}' \subsetneq \mathfrak{A}$, then $\mathfrak{J}$ is called a **maximal ideal**.*

Remarks

For a unital Banach algebra $\mathfrak{A}$:

1. *Every maximal ideal is closed.*
2. *Every proper ideal is contained in a maximal ideal.*

Definition

For $\mathfrak{A}$ a unital commutative Banach algebra, define

$$\mathcal{M}_{\mathfrak{A}} = \{\varphi : \mathfrak{A} \to \mathbb{C} : \varphi \text{ is a complex homomorphism}\}.$$

We call $\mathcal{M}_{\mathfrak{A}}$ the **maximal ideal space** *of $\mathfrak{A}$.*

Theorem

Let $\mathfrak{A}$ be a unital commutative Banach algebra.

1. *Every maximal ideal in $\mathfrak{A}$ is the kernel of some $\varphi \in \mathcal{M}_{\mathfrak{A}}$.*
2. *If $\varphi \in \mathcal{M}_{\mathfrak{A}}$, then $\ker \varphi$ is a maximal ideal in A.*
3. *$A \in \mathfrak{A}$ is invertible iff $\varphi(A) \neq 0$ for all $\varphi \in \mathcal{M}_{\mathfrak{A}}$.*
4. *$A \in \mathfrak{A}$ is invertible iff A lies in no proper ideal of $\mathfrak{A}$.*
5. *$\lambda \in \sigma(A)$ iff $\lambda = \varphi(A)$ for some $\varphi \in \mathcal{M}_{\mathfrak{A}}$.*

Proposition

If $\mathfrak{A}$ is a unital commutative C^*-algebra and A is self-adjoint, then $\sigma(A) \subset \mathbb{R}$.

Proof

We will show $\varphi(A) \in \mathbb{R}$ for all $\varphi \in \mathcal{M}_{\mathfrak{A}}$.

1. *For A self-adjoint in $\mathfrak{A}$, and $t \in \mathbb{R}$, we have*
$$\begin{aligned} |\varphi(A+itI)|^2 &\leq \|\varphi\|^2 \|A+itI\|^2 = \|A+itI\|^2 \\ &= \|(A+itI)^*(A+itI)\| = \|(A-itI)(A+itI)\| \\ &= \left\|A^2+t^2I^2\right\| \leq \|A\|^2 + t^2. \end{aligned}$$
2. *For $\varphi(A) = a + bi$, we have*
$$\|A\|^2 + t^2 \geq |\varphi(A+itI)|^2 = |\varphi(A)+it|^2 = a^2+b^2+2bt+t^2.$$
3. $\|A\|^2 \geq a^2+b^2+2bt$. *Suppose $b \neq 0$. By letting $t \to \pm\infty$, we see $\|A\|$ is unbounded for all self-adjoint A. I is self-adjoint and $\|I\| = 1$. So $b = 0$ and $\varphi(A) = a \in \mathbb{R}$.*